

# ON HERMITE HADAMARD-TYPE INEQUALITIES FOR STRONGLY LOG-CONVEX FUNCTIONS

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**ABSTRACT.** In this paper, the notation of strongly log-convex functions with respect to  $c > 0$  is introduced and versions of Hermite Hadamard-type inequalities for strongly logarithmic convex functions are established.

## 1. INTRODUCTION

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g., [8, p.137], [3]). These inequalities state that if  $f : I \rightarrow \mathbb{R}$  is a convex function on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ , then

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

The inequality (1.1) has evoked the interest of many mathematicians. Especially in the last three decades numerous generalizations, variants and extensions of this inequality have been obtained, to mention a few, see ([1]-[12]) and the references cited therein.

**Definition 1.** The function  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ , is said to be convex if the following inequality holds

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all  $x, y \in [a, b]$  and  $\lambda \in [0, 1]$ . We say that  $f$  is concave if  $(-f)$  is convex.

In [6], Pearce et. al. generalized this inequality to  $r$ -convex positive function  $f$  which defined on an interval  $[a, b]$ , for all  $x, y \in [a, b]$  and  $t \in [0, 1]$

$$f(tx + (1 - t)y) \leq \begin{cases} (t[f(x)]^r + (1 - t)[f(y)]^r)^{\frac{1}{r}}, & \text{if } r \neq 0 \\ [f(x)]^t [f(y)]^{1-t}, & \text{if } r = 0. \end{cases}$$

We have that 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions.

Recently, the generalizations of the Hermite-Hadamard's inequality to the integral power mean of a positive convex function on an interval  $[a, b]$ , and to that of a positive  $r$ -convex function on an interval  $[a, b]$  are obtained by Pearce and Pecaric, and others (see [6]-[12]).

A function  $f : I \rightarrow [0, \infty)$  is said to be log-convex or multiplicatively convex if  $\log t$  is convex, or, equivalently, if for all  $x, y \in I$  and  $t \in [0, 1]$  one has the inequality:

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$$(1.2) \quad f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}.$$

We note that if  $f$  and  $g$  are convex and  $g$  is increasing, then  $g \circ f$  is convex; moreover, since  $f = \exp(\log f)$ , it follows that a log-convex function is convex, but the converse may not necessarily be true [6]. This follows directly from (1.2) because, by the arithmetic-geometric mean inequality, we have

$$[f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1-t)f(y)$$

for all  $x, y \in I$  and  $t \in [0, 1]$ .

For some results related to this classical results, (see[3],[4],[9],[10]) and the references therein. Dragomir and Mond [3] proved the following Hermite-Hadamard type inequalities for the log-convex functions:

$$(1.3) \quad \begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a} \int_a^b \ln[f(x)] dx\right] \\ &\leq \frac{1}{b-a} \int_a^b G(f(x), f(a+b-x)) dx \\ &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq L(f(a), f(b)) \\ &\leq \frac{f(a) + f(b)}{2}, \end{aligned}$$

where  $G(p, q) = \sqrt{pq}$  is the geometric mean and  $L(p, q) = \frac{p-q}{\ln p - \ln q}$  ( $p \neq q$ ) is the logarithmic mean of the positive real numbers  $p, q$  (for  $p = q$ , we put  $L(p, q) = p$ ).

Recall also that a function  $f : I \rightarrow R$  is called strongly convex with modulus  $c > 0$ , if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

for all  $x, y \in I$  and  $t \in (0, 1)$ . Strongly convex functions have been introduced by Polyak in [13] and they play an important role in optimization theory and mathematical economics. Various properties and applicatins of them can be found in the literature see ([13]-[16]) and the references cited therein.

In this paper we introduce the notation of strongly logarithmic convex with respect to  $c > 0$  and versions of Hermite-Hadamard-type inequalities for strongly logarithmic convex with respect to  $c > 0$  are presented. This result generalizes the Hermite-Hadamard-type inequalities obtained in [3] for log-convex functions with  $c = 0$ .

## 2. MAIN RESULTS

We will say that a positive fuction  $f : I \rightarrow (0, \infty)$  is strongly log-convex with respect to  $c > 0$  if

$$f(\lambda x + (1-\lambda)y) \leq [f(x)]^\lambda [f(y)]^{1-\lambda} - c\lambda(1-\lambda)(x-y)^2$$

for all  $x, y \in I$  and  $\lambda \in (0, 1)$ . In particular, from the above definition, by the arithmetic-geometric mean inequality, we have

$$\begin{aligned}
 (2.1) \quad f(\lambda x + (1-\lambda)y) &\leq [f(x)]^\lambda [f(y)]^{1-\lambda} - c\lambda(1-\lambda)(x-y)^2 \\
 &\leq \lambda f(x) + (1-\lambda)f(y) - c\lambda(1-\lambda)(x-y)^2 \\
 &\leq \max\{f(x), f(y)\} - c\lambda(1-\lambda)(x-y)^2
 \end{aligned}$$

**Theorem 1.** *If a function  $f : I \rightarrow (0, \infty)$  be a strongly log-convex with respect to  $c > 0$  and Lebesgue integrable on  $I$ , we have*

$$\begin{aligned}
 (2.2) \quad f\left(\frac{a+b}{2}\right) + \frac{c(b-a)^2}{12} &\leq \frac{1}{b-a} \int_a^b G(f(x), f(a+b-x)) dx \\
 &\leq \frac{1}{b-a} \int_a^b f(x) dx \\
 &\leq L(f(a), f(b)) - \frac{c(b-a)^2}{6} \\
 &\leq \frac{f(a) + f(b)}{2} - \frac{c(b-a)^2}{6}
 \end{aligned}$$

for all  $a, b \in I$  with  $a < b$ .

*Proof.* From (2.1), we have

$$\begin{aligned}
 (2.3) \quad f(\lambda x + (1-\lambda)y) &\leq [f(x)]^\lambda [f(y)]^{1-\lambda} - c\lambda(1-\lambda)(x-y)^2 \\
 &\leq \lambda f(x) + (1-\lambda)f(y) - c\lambda(1-\lambda)(x-y)^2.
 \end{aligned}$$

Since  $f$  is a strongly log-convex function on  $I$ , we have for  $x, y \in I$  with  $\lambda = \frac{1}{2}$

$$\begin{aligned}
 (2.4) \quad f\left(\frac{x+y}{2}\right) &\leq \sqrt{f(x)f(y)} - \frac{c(x-y)^2}{4} \\
 &\leq \frac{f(x) + f(y)}{2} - \frac{c(x-y)^2}{4}
 \end{aligned}$$

i.e., with  $x = ta + (1-t)b$ ,  $y = (1-t)a + tb$ ,

$$\begin{aligned}
 (2.5) \quad f\left(\frac{a+b}{2}\right) &\leq \sqrt{f(ta + (1-t)b)f((1-t)a + tb)} - \frac{c(b-a)^2(1-2t)^2}{4} \\
 &\leq f(ta + (1-t)b) + f((1-t)a + tb) - \frac{c(b-a)^2(1-2t)^2}{4}.
 \end{aligned}$$

Integrating the inequality (2.5) with respect to  $t$  over  $(0, 1)$ , we obtain

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b \sqrt{f(x)f(a+b-x)} dx - \frac{c(b-a)^2}{12} \\
 &\leq \frac{1}{b-a} \int_a^b A(f(x), f(a+b-x)) dx - \frac{c(b-a)^2}{12},
 \end{aligned}$$

and so for  $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$ ,

$$(2.6) \quad f\left(\frac{a+b}{2}\right) + \frac{c(b-a)^2}{12} \leq \frac{1}{b-a} \int_a^b G(f(x), f(a+b-x)) dx$$

$$\leq \frac{1}{b-a} \int_a^b f(x) dx.$$

Since  $f$  is a strongly log-convex function on  $I$ , for  $x = a$  and  $y = b$ , we write

$$(2.7) \quad f(ta + (1-t)b) \leq [f(a)]^t [f(b)]^{1-t} - ct(1-t)(a-b)^2$$

$$\leq tf(a) + (1-t)f(b) - ct(1-t)(a-b)^2.$$

Integrating the inequality (2.7) with respect to  $t$  over  $(0, 1)$ , we obtain,

$$\frac{1}{b-a} \int_a^b f(x) dx \leq f(b) \int_0^1 \left[ \frac{f(a)}{f(b)} \right]^t dt - c(b-a)^2 \int_0^1 t(1-t) dt$$

$$\leq f(a) \int_0^1 t dt + f(b) \int_0^1 (1-t) dt - c(b-a)^2 \int_0^1 t(1-t) dt,$$

and so

$$(2.8) \quad \frac{1}{b-a} \int_a^b f(x) dx \leq L(f(a), f(b)) - \frac{c(b-a)^2}{6} \leq \frac{f(a) + f(b)}{2} - \frac{c(b-a)^2}{6}.$$

Thus, from (2.6) and (2.8), we obtain the inequality of (2.2). This completes to proof.  $\square$

**Theorem 2.** Let a function  $f : I \rightarrow [0, \infty)$  be a strongly log-convex with respect to  $c > 0$  and Lebesgue integrable on  $I$ , then the following inequality holds:

$$(2.9) \quad \frac{1}{b-a} \int_a^b f(x) f(a+b-x) dx \leq f(a) f(b) + \frac{c^2(b-a)^4}{30}$$

$$- \frac{4c(b-a)^2}{[\ln(f(b) - f(a))]^2} [A(f(a), f(b)) + L(f(a), f(b))]$$

for all  $a, b \in I$  with  $a < b$ .

*Proof.* Since  $f$  is strongly log-convex with respect to  $c > 0$ , we have that for all  $t \in (0, 1)$

$$(2.10) \quad f(ta + (1-t)b) \leq [f(a)]^t [f(b)]^{1-t} - ct(1-t)(b-a)^2$$

$$\leq tf(a) + (1-t)f(b) - ct(1-t)(b-a)^2$$

and

$$\begin{aligned}
 f((1-t)a + tb) &\leq [f(a)]^{1-t} [f(b)]^t - ct(1-t)(b-a)^2 \\
 (2.11) \qquad &\leq (1-t)f(a) + tf(b) - ct(1-t)(b-a)^2.
 \end{aligned}$$

Multiplying both sides of (2.10) by (2.11), it follows that

$$\begin{aligned}
 f(ta + (1-t)b)f((1-t)a + tb) &\leq f(a)f(b) + c^2(b-a)^4 t^2(1-t)^2 \\
 (2.12) \qquad &\quad -c(b-a)^2 t(1-t) \left( f(b) \left[ \frac{f(a)}{f(b)} \right]^t + f(a) \left[ \frac{f(b)}{f(a)} \right]^t \right).
 \end{aligned}$$

Integrating the inequality (2.12) with respect to  $t$  over  $(0, 1)$ , we obtain

$$\begin{aligned}
 \int_a^b f(ta + (1-t)b)f((1-t)a + tb) dt &\leq \int_0^1 f(a)f(b) dt + c^2(b-a)^4 \int_0^1 t^2(1-t)^2 dt \\
 (2.13) \qquad &\quad -c(b-a)^2 f(b) \int_0^1 t(1-t) \left[ \frac{f(a)}{f(b)} \right]^t dt - c(b-a)^2 f(a) \int_0^1 t(1-t) \left[ \frac{f(b)}{f(a)} \right]^t dt \\
 &= \int_0^1 f(a)f(b) dt + c^2(b-a)^4 \int_0^1 t^2(1-t)^2 dt - c(b-a)^2 f(b) I_1 - c(b-a)^2 f(a) I_2.
 \end{aligned}$$

Integrating by parts for  $I_1$  and  $I_2$  integrals, we obtain

$$\begin{aligned}
 I_1 &= \int_0^1 t(1-t) \left[ \frac{f(a)}{f(b)} \right]^t dt \\
 &= t(1-t) \frac{1}{\ln \left[ \frac{f(a)}{f(b)} \right]} \left[ \frac{f(a)}{f(b)} \right]^t \Big|_0^1 - \frac{1}{\ln \left[ \frac{f(a)}{f(b)} \right]} \int_0^1 (1-2t) \left[ \frac{f(a)}{f(b)} \right]^t dt \\
 (2.14) \qquad &= -\frac{1}{\ln \left[ \frac{f(a)}{f(b)} \right]} \left[ (1-2t) \frac{1}{\ln \left[ \frac{f(a)}{f(b)} \right]} \left[ \frac{f(a)}{f(b)} \right]^t \Big|_0^1 + \frac{2}{\ln \left[ \frac{f(a)}{f(b)} \right]} \int_0^1 \left[ \frac{f(a)}{f(b)} \right]^t dt \right] \\
 &= \frac{1}{f(b)} \frac{f(a) + f(b)}{[\ln(f(a) - f(b))]^2} + \frac{2f(a) - 2f(b)}{[\ln(f(a) - f(b))]^2},
 \end{aligned}$$

and similarly we get,

$$\begin{aligned}
 (2.15) \qquad I_2 &= \int_0^1 t(1-t) \left[ \frac{f(b)}{f(a)} \right]^t dt \\
 &= \frac{1}{f(a)} \frac{f(a) + f(b)}{[\ln(f(b) - f(a))]^2} + \frac{2f(b) - 2f(a)}{[\ln(f(b) - f(a))]^2}.
 \end{aligned}$$

Putting (2.14) and (2.15) in (2.13), and if we change the variable  $x := ta + (1 - t)b$ ,  $t \in (0, 1)$ , we get the required inequality in (2.9). This proves the theorem.  $\square$

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